Student Name: ________________________

Teacher: ________________________

2016

HSC ASSESSMENT

TASK4 ~ TRIAL HSC

Mathematics Extension 2

Examiners

Mr J. Dillon, Mr G. Huxley and Mr G. Rawson

General Instructions

- Reading time - 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- Diagrams may be drawn in pencil.
- Board-approved calculators and mathematical templates may be used.
- Answer Section 1 on the separate answer sheet provided.
- Show all necessary working in Questions 11 – 16.
- Start each of Questions 11 – 16 in a separate answer booklet.
- Put your name on each booklet.
- This question booklet is not to be removed from the examination room

Total marks - 100

Section I

10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.

Section II

90 marks

- Attempt Questions 11 – 16. Each of these six questions are worth 15 marks.
- Allow about 2 hours 45 minutes for this section.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1. Let $z = 4 - i$. What is the value of $\overline{iz}$?

   (A) $-1-4i$

   (B) $-1+4i$

   (C) $1-4i$

   (D) $1+4i$

2. If $z = 1+2i$ and $w = 3-i$, which expression gives $z - \overline{w}$?

   (A) $3i-2$

   (B) $4+3i$

   (C) $i-2$

   (D) $4+i$

3. Which expression is equal to $\int 3\sqrt{x}\ln x \, dx$?

   (A) $2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + c$

   (B) $2x\sqrt{x}\left(\ln x + \frac{2}{3}\right) + c$

   (C) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x - 1\right) + c$

   (D) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x + 1\right) + c$

4. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6

(B) 3

(C) -1

(D) -6

5. What is the eccentricity of the hyperbola with the equation $\frac{x^2}{3} - \frac{y^2}{4} = 1$?

(A) $1 + \frac{2}{\sqrt{3}}$

(B) $\sqrt{\frac{7}{3}}$

(C) $\frac{\sqrt{7}}{3}$

(D) $\frac{5}{3}$

6. If a, b, c, d and e are real numbers and $a \neq 0$, which of the following statements is correct?

(A) the polynomial equation $ax^7 + bx^5 + cx^3 + dx + e = 0$ has only one real root

(B) the polynomial equation $ax^7 + bx^5 + cx^3 + dx + e = 0$ has at least one real root

(C) the polynomial equation $ax^7 + bx^5 + cx^3 + dx + e = 0$ has an odd number of non-real roots

(D) the polynomial equation $ax^7 + bx^5 + cx^3 + dx + e = 0$ has no real roots

7. What is the number of asymptotes on the graph of $y = \frac{2x^3}{x^2 - 1}$?

(A) 1

(B) 2

(C) 3

(D) 4

8. At how many points do the graphs of $y=|x|$ and $y=|x^2-4|$ intersect?

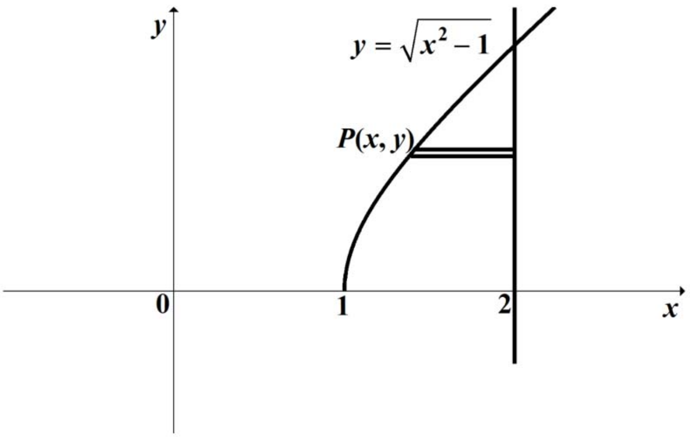
(A) 0

(B) 1

(C) 2

(D) 4

9.



The region bounded by the x-axis, the curve $y=\sqrt{x^2-1}$ and the line $x = 2$ is rotated about the y-axis.

The slice at $P(x, y)$ on the curve is perpendicular to the axis of rotation.

What is the volume δV of the annular slice formed?

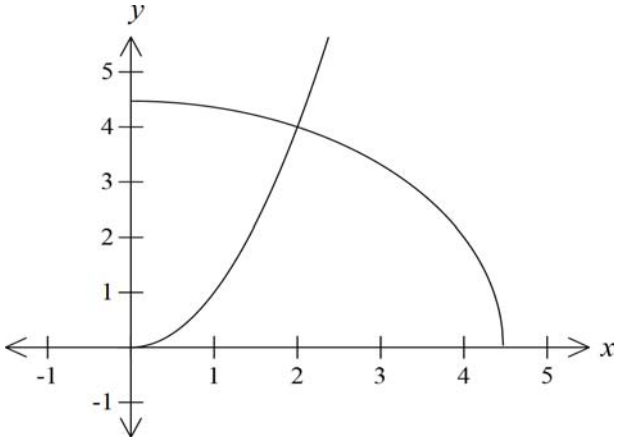
(A) $\pi\left(3-y^2\right)\delta y$

(B) $\pi\left(4-\left(y^2+1\right)^2\right)\delta y$

(C) $\pi\left(4-\left(x^2-1\right)\right)\delta x$

(D) $\pi\left(2-\sqrt{x^2-1}\right)\delta x$

10. What is the correct expression for volume of the solid formed when the region bounded by the curves $y = x^2$, $y = \sqrt{20 - x^2}$ and the y-axis is rotated about the y-axis?



(A) $V = \int_0^2 2\pi\left(\sqrt{20 - x^2} - x^2\right)dx$

(B) $V = \int_0^2 2\pi x\left(\sqrt{20 - x^2} - x^2\right)dx$

(C) $V = \int_0^2 2\pi\left(x^2 - \sqrt{20 - x^2}\right)dx$

(D) $V = \int_0^2 2\pi x\left(x^2 - \sqrt{20 - x^2}\right)dx$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

***Answer each question in a new answer booklet*.**

All necessary working should be shown in every question.

Question 11 ***Answer this question in a new answer booklet***

(a) Let $z = \cos\theta + i\sin\theta$ where θ is real.

(i) Use De Moivre's theorem to show that $\frac{1}{z} = \cos\theta - i\sin\theta$. **1**

(ii) Hence, or otherwise, find $z^n - \frac{1}{z^n}$ **1**

(b) Let $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1+2i}$, where a and b are real numbers.
What is the value of a and b, if $z_1 + z_2 = 1$? **2**

(c) Let w be a non-real cube root of unity.

(i) Show that $1 + w + w^2 = 0$ **1**

(ii) Hence or otherwise, evaluate: $\frac{1}{1+w} + \frac{1}{1+w^2}$ **1**

(d) Sketch the locus of points on an Argand diagram that satisfy:

$$\arg\left(\frac{z-2}{z+2i}\right) = \frac{\pi}{2}$$ **2**

(e) (i) Show that $z\overline{z} = |z|^2$ for any complex number z. **1**

(ii) A sequence of complex numbers z_n is given by the rule
$z_1 = w$ and $z_n = v\overline{z}_{n-1}$ where w is a given complex number and
v is a complex number with modulus 1. Show that $z_3 = w$. **2**

Question 11 continues on the next page ….

(f) Solve simultaneously by graphing both equations on an Argand Diagram and expressing the point of intersection in the form $x + i\,y$:

$|z+2|=2$ and $\arg z = \dfrac{3\pi}{4}$ **4**

Question 12 ***Answer this question in a new answer booklet***

(a) Find $\int \cos x \sin^4 x \, dx$. **1**

(b) Find $\int \frac{dx}{x^2 - 4x + 8}$. **2**

(c) Use the substitution $u = x - 2$ to find the exact value of $\int_1^3 x(x-2)^5 \, dx$. **3**

(d) (i) Find the values of A, B and C so that **2**

$$\frac{5}{(x^2+4)(x+1)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x+1}.$$

(ii) Hence find $\int \frac{5}{(x^2+4)(x+1)} \, dx$. **3**

(e) (i) If $I_n = \int_1^e x(\ln x)^n \, dx$ for $n = 0,1,2,3,\ldots$ use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n = 1,2,3,\ldots$ **2**

(ii) Hence find the value of I_2. **2**

Question 13 ***Answer this question in a new answer booklet***

(a) If α, β and γ are the roots of the equation $x^3 - 3x^2 + 2x - 1 = 0$, find:

(i) $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **3**

(iii) the equation whose roots are α^{-1}, β^{-1} and γ^{-1} **2**

(b) The three roots of the equation $8x^3 - 36x^2 + 38x - 3 = 0$ are in arithmetic sequence. Find the roots of the equation. **3**

(c) An ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

(i) Prove that the tangent to the ellipse at $P(4\cos\theta, 3\sin\theta)$ has equation

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$$ **3**

(ii) The ellipse meets the *y*-axis at *B* and *B*'.
The tangents at *B* and *B*' meet the tangent at *P* at the points *Q* and *Q*'.
Find $BQ \times B'Q'$. **3**

Question 14 ***Answer this question in a new answer booklet***

(a) The zeros of the equation $x^4 + 4x^3 - mx - b = 0$ are α, α, β and β.

Illustrate how this can be shown on a graph, which includes $y = x^4 + 4x^3$.
You do not have to find m, b, α or β. **2**

(b) Consider the function $f(x) = (3-x)(x+1)$. On separate axes, sketch, showing the important features, the graphs of:

(i) $y = f(x)$ **1**

(ii) $y = |f(x)|$ **1**

(iii) $y = f|(x)|$ **1**

(iv) $|y| = f(x)$ **1**

(v) $y^2 = f(x)$ **2**

(vi) $y = \log_2 [f(x)]$ **2**

(c) If $x^2 + y^2 + xy = 3$,

(i) Find $\dfrac{dy}{dx}$ **2**

(ii) Sketch, showing the critical points and stationary points, the graph of:

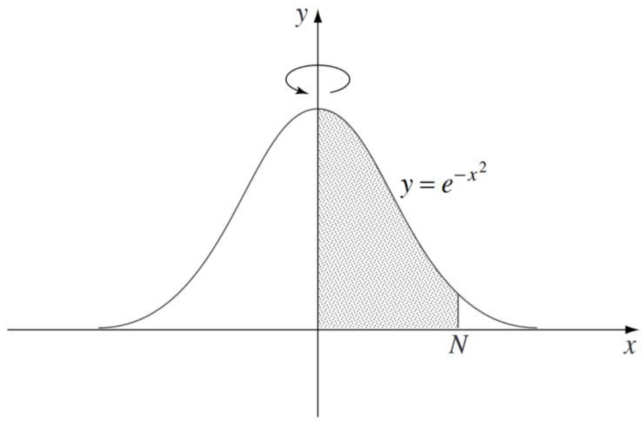
$x^2 + y^2 + xy = 3$ **3**

Question 15 ***Answer this question in a new answer booklet***

(a) The area between the coordinate axes and the line $2x + 3y = 6$ is rotated about the line $y = 3$.
By taking slices perpendicular to the axis of rotation, show that the volume of the solid formed is given by

$$V = \pi \int_0^3 \left(8 - \frac{4x}{3} - \frac{4x^2}{9} \right) dx$$ **2**

(b) The shaded region between the curve $y = e^{-x^2}$, the x-axis, and the lines $x = 0$ and $x = N$, where $N > 0$, is rotated about the y-axis to form a solid of revolution.

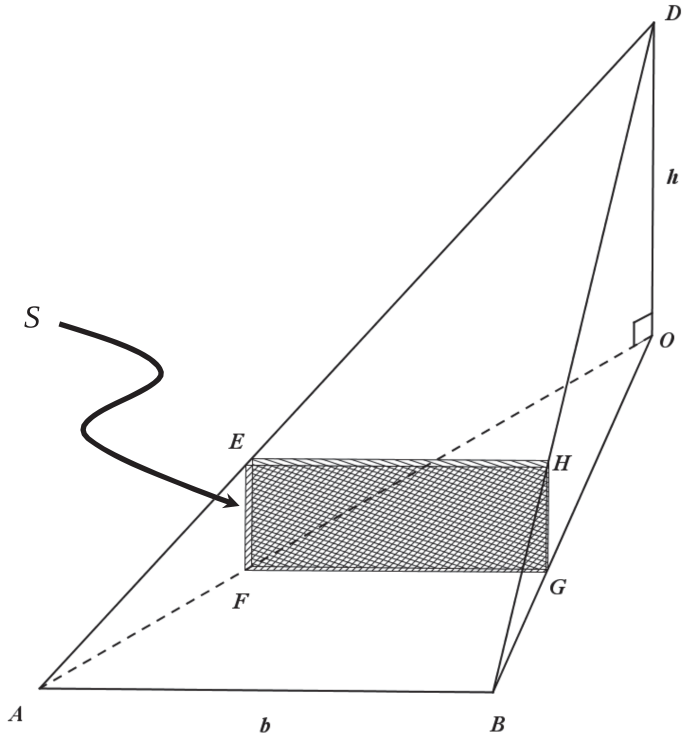


(i) Use the method of cylindrical shells to find the volume of this solid in terms of N. **3**

(ii) What is the limiting value of this volume as $N \to \infty$? **1**

Question 15 continues on the next page ….

(c)



Let OAB be an isosceles triangle, with $OA = OB = r$ units, and $AB = b$ units.
Let $OABD$ be a triangular pyramid with height $OD = h$ units and OD perpendicular to the plane OAB as in the diagram above.

Consider a slice, S, of the pyramid of width δa as shown at $EFGH$ in the diagram.
The slice S is perpendicular to the plane OAB at FG, with $FG \parallel AB$ and $BG = a$ units.
Note also, that $GH \parallel$ OD.

(i) Show that the volume of S is $\frac{b(r-a)}{r}\left(\frac{ah}{r}\right)\delta a$ when δa is small. **3**

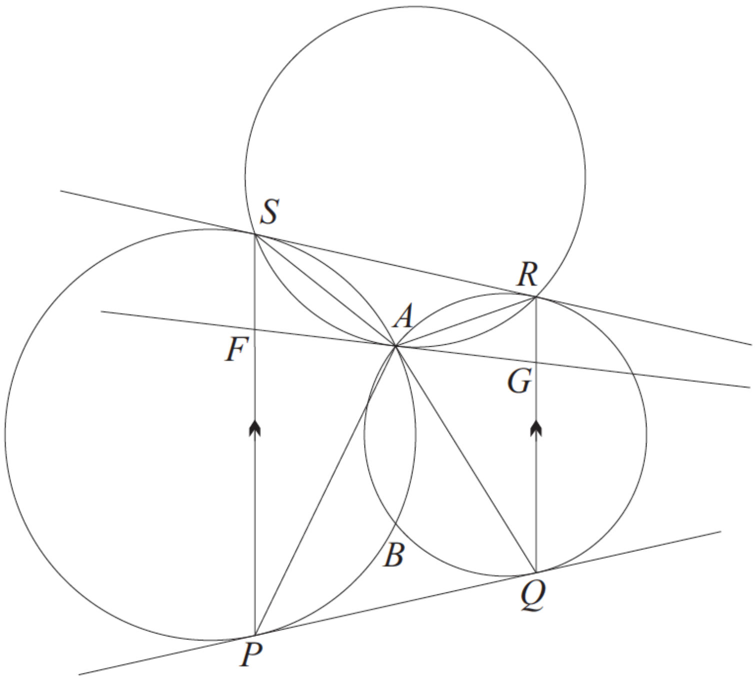
(ii) Hence, show that the pyramid $DOAB$ has a volume of $\frac{1}{6}hbr$. **2**

(iii) Suppose now that $\angle AOB = \frac{2\pi}{n}$ and that n identical pyramids $DOAB$ are arranged about O as the centre, with common vertical axis OD to form a solid C.
Show that the volume V_n of C is given by $V_n = \frac{1}{3}r^2hn\sin\frac{\pi}{n}$. **2**

(iv) Note that when n is large, C approximates a right circular cone.
Hence, find $\lim_{n\to\infty} V_n$ and verify that a right circular cone of radius r and height h has a volume $\frac{1}{3}\pi r^2 h$ **2**

Question 16 ***Answer this question in a new answer booklet***

(a)



In the diagram above, two circles of differing radii intersect at A and B. The lines PQ and RS are the common tangents with $PS \parallel QR$.
A third circle passes through the points S, A and R.
The tangent to this circle at A meets the parallel lines at F and G.
Let $\angle RAG = \alpha$, $\angle AGR = \beta$ and $\angle GRA = \gamma$.

NOTE: You do not need to copy the diagram above. It has been reproduced for you on a tear – off sheet at the end of this paper. Insert this sheet into your answer booklet for Question 16.

(i) Show that $\angle SPA = \alpha$ **2**

(ii) Hence, prove that FG is also a tangent to the circle which passes through the points A, P and Q. **3**

(b) ΔABC has sides of length a, b and c.
If $a^2 + b^2 + c^2 = ab + bc + ca$ show that ΔABC is an equilateral triangle. **3**

Question 16 continues on the next page ….

(c) (i) Use the binomial theorem $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$ to show that **1**

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} \times \frac{1}{k!}$$

(ii) Hence, show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$ **2**

(iii) Prove by induction that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ when $n \geq 3$ and n is an integer. **3**

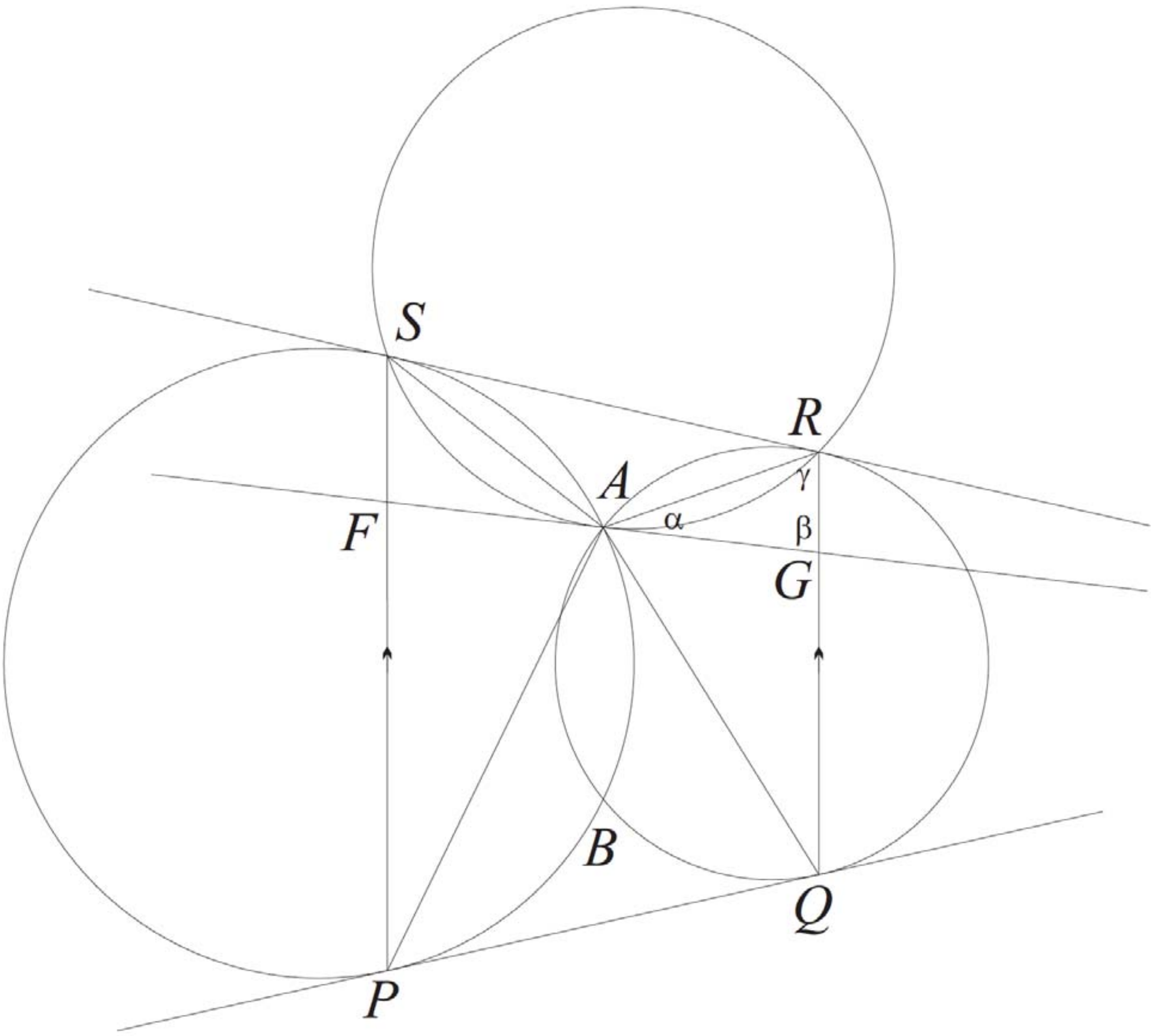
(iv) Hence, show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 3$. **1**

STUDENT NAME: ________________________________

DETACH THIS AND INCLUDE IT IN YOUR ANSWER BOOKLET FOR QUESTION 16

Question 16

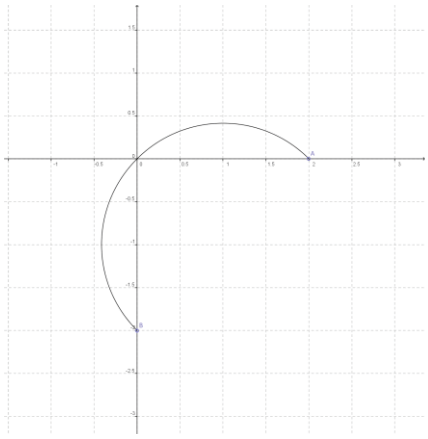
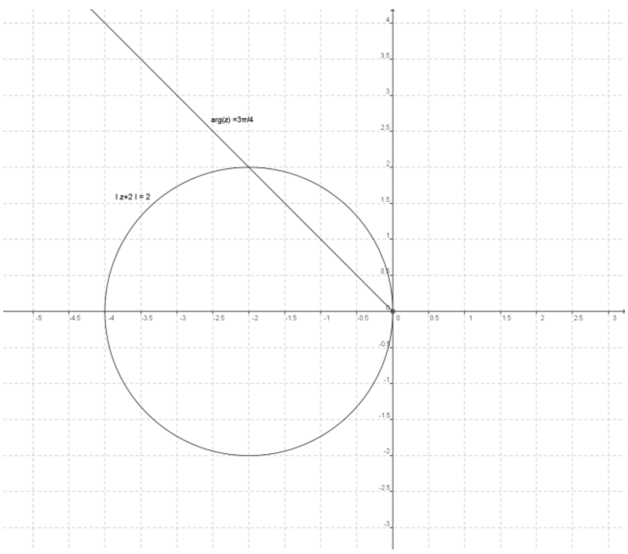
(a)



Question 11: Outcomes Addressed in this Question:

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

Outcome	Solutions	Marking Guidelines
E3 **(a)**	(i) $z^{-1}=(\cos\theta+i\sin\theta)^{-1}=(\cos(-\theta)+i\sin(-\theta))$ $=\cos\theta-i\sin\theta$	(a) (i) **1 mark:** Correct "show" of Demoivre
	(ii) $z^n-\dfrac{1}{z^n}=\cos n\theta+i\sin\theta-(\cos n\theta-i\sin n\theta)$ $=2i\sin n\theta$	(ii) **1 mark:** correct answer.
(b)	(i) $1=\dfrac{a+2ai+b+bi}{(1+i)(1+2i)}$ $\therefore\ a+b=-1 \qquad 2a+b=3$ $\therefore \qquad a=4 \quad b=-5$	(b) (i) **2 marks:** Correct solution. **1 mark:** Significant progress.
(c)	(i) w is a cube root of unity, so $w^3-1=0$ $(w-1)(w^2+w+1)=0 \qquad w$ not real, so $w-1\neq 0$ $\therefore w^2+w+1=0$	(c) (i) **1 mark:** correct solution including reason.
	(ii) $\dfrac{1}{1+w}+\dfrac{1}{1+w^2}=\dfrac{1}{-w}-\dfrac{1}{w}$ $=\dfrac{-w-w^2}{w^3}$ $=1$	(ii) **1 mark:** Correct solution. There are several correct methods.

(d)	$\arg\left(\dfrac{z-2}{z+2i}\right)=\dfrac{\pi}{2}$ 	**<u>2 marks:</u>** Correct semi-circle, with diameter end-points excluded. (2,0) and (0, -2). Needs to pass through origin. **<u>1 mark:</u>** Significant progress.
(e)	(i) $z\bar{z}=x^2-(iy)^2$ $=x^2+y^2$ $=\lvert z\rvert^2$ (ii) $z_1=w \qquad z_2=v\bar{w} \qquad z_3=v\bar{v}w$ But $v\bar{v}=\lvert v\rvert^2=1$ from (i) and because *v* has modulus 1. $\therefore z_3=w$	**<u>(e) (i) 1 mark:</u>** Show that difference of two squares becomes a sum. **<u>(ii) 2 marks:</u>** Correct solution including explanation in regards to part (i). **<u>1 mark:</u>** Partial progress.
(f)	 Pt of intersection: $z=-2+2i$	**<u>(f) 4 marks:</u>** Correct solution, including circle, line, excluded point, value of *z*. **<u>3 marks:</u>:** One element of solution omitted. **<u>2 marks:</u>** Significant progress. **<u>1 mark:</u>** Some relevant progress.

Outcome Addressed in this Question

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

Outcomes	Solutions	Marking Guidelines
(a)	$\int \cos x \sin^4 x\, dx = \frac{\sin^5 x}{5} + c$	**Award 1** for correct answer
(b)	$\int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2+2^2}$ $= \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + c$	**Award 2** for correct solution **Award 1** for substantial progress towards solution
(c)	$I = \int_1^3 x(x-2)^5\, dx$ $\quad u = x-2 \to du = dx$ $x = 1, u = -1$ $x = 3, u = 1$ $\therefore I = \int_{-1}^{1} (u+2)u^5\, du$ $= \int_{-1}^{1} \left(u^6 + 2u^5\right) du$ $= \left[\frac{u^7}{7} + \frac{2u^6}{6}\right]_{-1}^{1}$ $= \left(\frac{1}{7} + \frac{1}{3}\right) - \left(-\frac{1}{7} + \frac{1}{3}\right)$ $= \frac{2}{7}$	**Award 3** for correct answer. **Award 2** for significant progress towards solution **Award 1** for limited progress towards solution
(d) (i)	$5 \equiv (Ax+B)(x+1) + C\left(x^2+4\right)$ Let $x = -1$, $\quad 5 = 5C \to C = 1$ Let $x = 0$, $\quad 5 = B + 4C \to B = 1$ Let $x = 1$, $\quad 5 = 2(A+B) + 5C \to A = -1$	**Award 2** for correct values of *A*, *B* and *C* **Award 1** for substantial progress towards solution
(ii)	$I = \int \frac{5}{\left(x^2+4\right)(x+1)}\, dx$ $= \int \frac{-x+1}{x^2+4} + \frac{1}{x+1}\, dx$ $= \int \frac{-x}{x^2+4} + \frac{1}{x^2+4} + \frac{1}{x+1}\, dx$ $= -\frac{1}{2}\ln\left\|x^2+4\right\| + \frac{1}{2}\tan^{-1}\frac{x}{2} + \ln\left\|x+1\right\|$	**Award 3** for correct answer. **Award 2** for significant progress towards solution **Award 1** for limited progress towards solution

(d) (i)	$I_n = \int_1^e x(\ln x)^n \, dx \quad u = (\ln x)^n \quad \frac{dv}{dx} = x$ $\frac{du}{dx} = \frac{n}{x}(\ln x)^{n-1} \quad v = \frac{x^2}{2}$ $\therefore I_n = \left[\frac{x^2}{2}.(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2}.\frac{n}{x}(\ln x)^{n-1} \, dx$ $= \frac{e^2}{2}.(\ln e)^n - \frac{1^2}{2}.(\ln 1)^n - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$ $= \frac{e^2}{2} - \frac{n}{2}.I_{n-1}$	**Award 2** for correct solution **Award 1** for substantial progress towards solution
(ii)	$I_2 = \frac{e^2}{2} - \frac{2}{2}I_1$ $= \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2}I_0\right)$ $= \frac{1}{2}\int_1^e x \, dx$ $= \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$ $= \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{e^2 - 1}{4}$	**Award 2** for correct solution **Award 1** for substantial progress towards solution

Year 12	Mathematics Extension 2	TRIAL - 2016 HSC
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Part / Outcome	Solutions	Marking Guidelines
(a)	(i) $x^3-3x^2+2x-1=0$ $\alpha+\beta+\gamma=-\frac{b}{a}=3$ $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}=2$	<u>1 mark</u> : correct solution
	(ii) $\alpha^3-3\alpha^2+2\alpha-1=0$ $\beta^3-3\beta^2+2\beta-1=0$ $\gamma^3-3\gamma^2+2\gamma-1=0$ so, $\alpha^3+\beta^3+\gamma^3=3(\alpha^2+\beta^2+\gamma^2)-2(\alpha+\beta+\gamma)+3$ $=3(\alpha^2+\beta^2+\gamma^2)-2(3)+3$ $=3\left[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)\right]-3$ $=3\left[3^2-2(2)\right]-3$ $=12$	<u>3 marks</u> : correct solution <u>2 marks</u> : substantially correct solution <u>1 mark</u> : progress towards correct solution
	(iii) $\left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)^2+2\left(\frac{1}{x}\right)-1=0$ $\frac{1}{x^3}-\frac{3}{x^2}+\frac{2}{x}-1=0$ $1-3x+2x^2-x^3=0$ ie $x^3-2x^2+3x-1=0$	<u>2 marks</u> : correct solution <u>1 mark</u> : substantially correct solution
(b)	$8x^3-36x^2+38x-3=0$ roots in AP $\rightarrow$ $a-d,\ a,\ a+d$ $\alpha+\beta+\gamma=-\frac{b}{a}$ $3a=\frac{36}{8}$ $a=\frac{3}{2}$ $\alpha\beta\gamma=-\frac{d}{a}$ $a(a^2-d^2)=\frac{3}{8}$ $\frac{3}{2}\left(\left(\frac{3}{2}\right)^2-d^2\right)=\frac{3}{8}$ $\frac{9}{4}-d^2=\frac{1}{4}$ $d^2=2$ $d=\pm\sqrt{2}$ $\therefore$ the roots are $\frac{3}{2}-\sqrt{2}$, $\frac{3}{2}$, $\frac{3}{2}+\sqrt{2}$	<u>3 marks</u> : correct solution <u>2 marks</u> : substantially correct solution <u>1 mark</u> : progress towards correct solution

Question 13 coninued…

(c)

(i) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1 \qquad \rightarrow \qquad a=4,\ b=3$

$$\frac{2x}{16}+\frac{2y}{9}.\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{9x}{16y}$$

$$\text{so, } m=-\frac{3\cos\theta}{4\sin\theta} \quad \text{at } P\left(4\cos\theta,\ 3\sin\theta\right)$$

eq'n of tangent is $y-3\sin\theta=-\dfrac{3\cos\theta}{4\sin\theta}\left(x-4\cos\theta\right)$

$$4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$$

$$3x\cos\theta+4y\sin\theta=12\left(\sin^2\theta+\cos^2\theta\right)$$

$$3x\cos\theta+4y\sin\theta=12$$

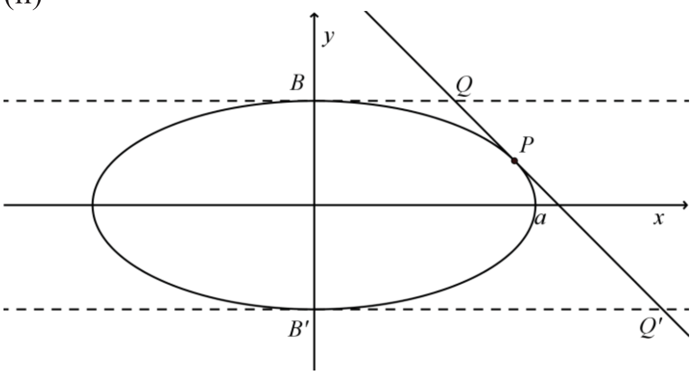
$$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=12$$

<u>3 marks</u> : correct solution

<u>2 marks</u> : substantially correct solution

<u>1 mark</u> : progress towards correct solution

(ii)



At Q: $y=3 \quad \rightarrow \quad x=\dfrac{4\left(1-\sin\theta\right)}{\cos\theta}$

At Q': $y=-3 \quad \rightarrow \quad x=\dfrac{4\left(1+\sin\theta\right)}{\cos\theta}$

so, $BQ\times BQ'=\dfrac{4\left(1-\sin\theta\right)}{\cos\theta}\times\dfrac{4\left(1+\sin\theta\right)}{\cos\theta}$

$$=\frac{16\left(1-\sin^2\theta\right)}{\cos^2\theta}$$

$$=\frac{16\cos^2\theta}{\cos^2\theta}$$

$$=16$$

<u>3 marks</u> : correct solution

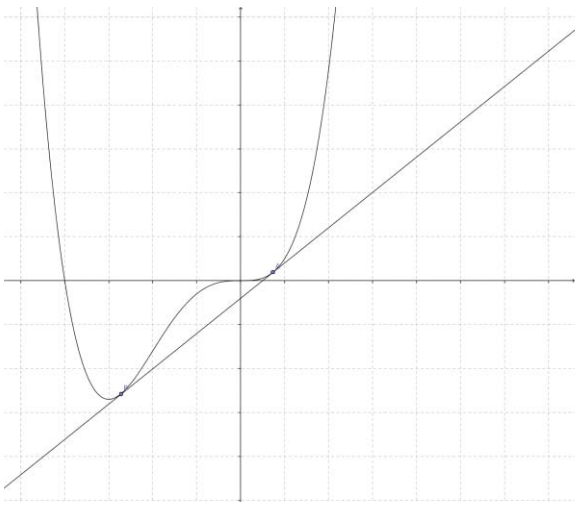
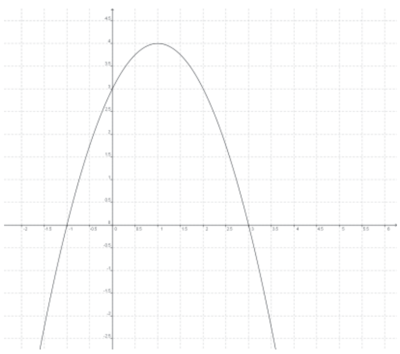
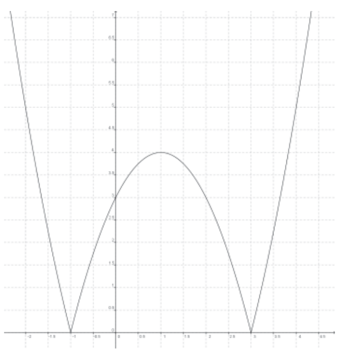
<u>2 marks</u> : substantially correct solution

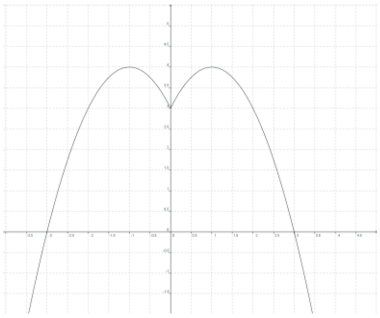
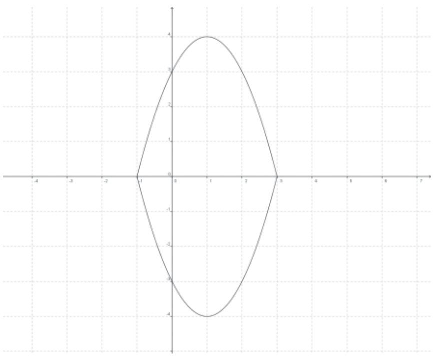
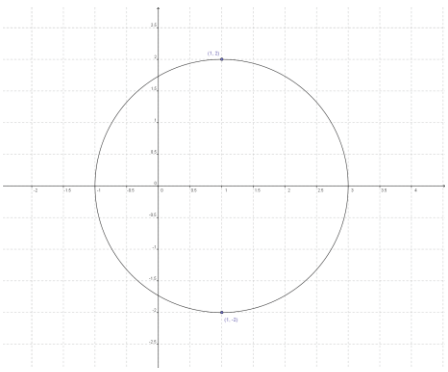
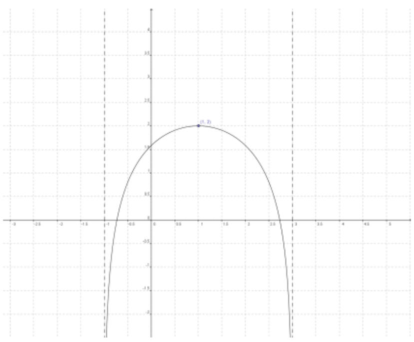
<u>1 mark</u> : progress towards correct solution

Mathematics Extension 2 Solutions and Marking Guidelines **Trial Exam 2016**

Question 14: Outcomes Addressed in this Question:

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

Outcome	Solutions	Marking Guidelines
E6 **(a)**	If $x^4+4x^3-mx-b=0$ Then $x^4+4x^3=mx+b$ We are told this equation has 2 double roots. 	(a) **2 marks:** Correct representation of both components of the sketch. **1 mark:** Partially correct.
(b)	(i)  (ii) 	(b) (i) **1 mark:** Correct parabola (ii) **1 mark:** correct sketch, including showing that the arms are concave up.

	(iii) 	(iii) **1 mark:** Correct reflection of the RHS of (i). .
	(iv) 	**(iv)1 mark:** Correct reflection of upper part of (i) only..
	(v) 	**(v) 2 marks:** Correct diagram (circle, centre (1,0) radius 2) with maximum and minimum turning points indicated. **1 mark:** Partially correct.
	(vi) 	**(vi) 2 marks:** Correct diagram, including maximum turning point, asymptotes at x=-1 and x = 3 **1 mark:** Partially correct.

(c)

(c) 2 marks: Correct solution

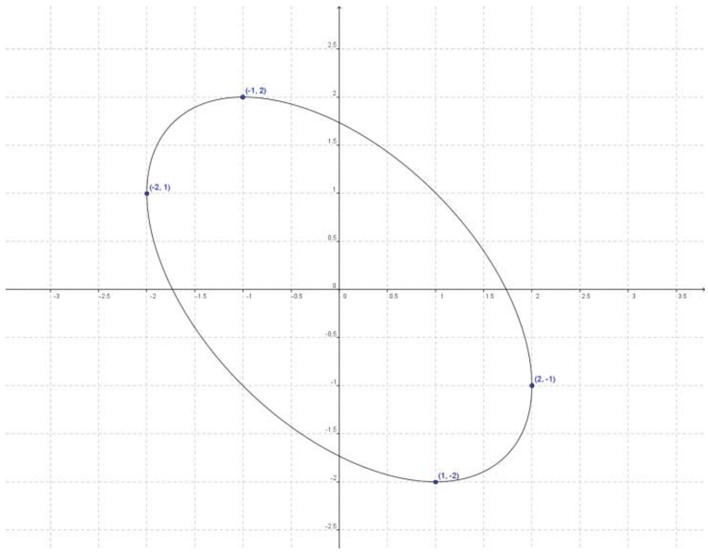
1 mark: partially correct.

(i)

$$2x+2y\frac{dy}{dx}+x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

(ii)



(iii) 3 marks: Correct solution and diagram, including stationary and critical points.

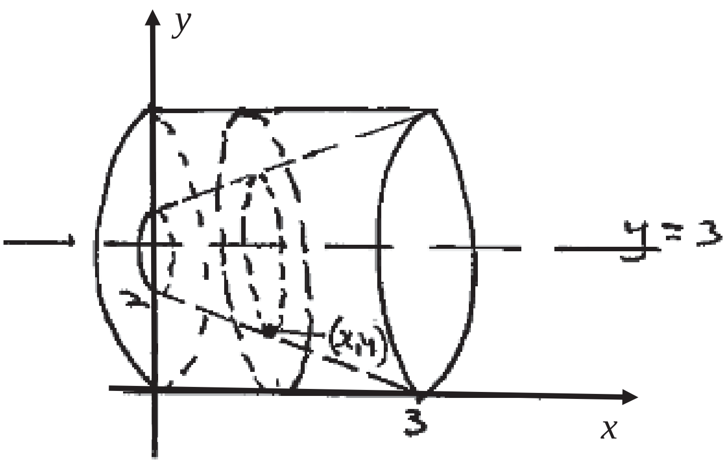
2 marks: Significant progress.

1 mark: Some relevant progress.

Year 12	Mathematics Extension 2	TRIAL - 2016 HSC
Question No. 15	Solutions and Marking Guidelines	

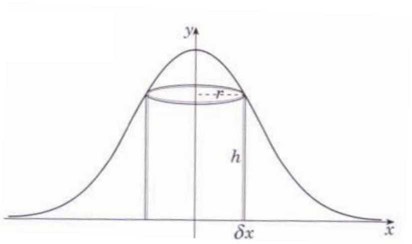
Outcomes Addressed in this Question

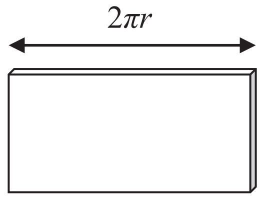
E7 - uses the techniques of slicing and cylindrical shells to determine volumes

Part / Outcome	Solutions	Marking Guidelines
(a)	 $A=\pi\left(R^2-r^2\right)$ $A(y)=\pi\left(3^2-(3-y)^2\right)$ $=\pi\left(6y-y^2\right)$ $A(x)=\pi\left[6\left(2-\frac{2x}{3}\right)-\left(2-\frac{2x}{3}\right)^2\right]$ $=\pi\left[12-4x-\left(4-\frac{8x}{3}+\frac{4x^2}{9}\right)\right]$ $=\pi\left(8-\frac{4x}{3}-\frac{4x^2}{9}\right)$ $\delta V=\pi\left(8-\frac{4x}{3}-\frac{4x^2}{9}\right)\delta x$ $V=\lim_{\delta x\to 0}\sum_{x=0}^{3}\pi\left(8-\frac{4x}{3}-\frac{4x^2}{9}\right)\delta x$ $=\pi\int_0^3\left(8-\frac{4x}{3}-\frac{4x^2}{9}\right)dx$ $2x+3y=6$ $3y=6-2x$ $y=2-\frac{2x}{3}$	2 marks : correct solution 1 mark : substantially correct solution

Question 15 continued…

(b) (i)





$$r = x,\ h = e^{-x^2}$$

$$\begin{aligned} A &= 2\pi rh = 2\pi xy \\ A(x) &= 2\pi x e^{-x^2} \\ \delta V &= 2\pi x e^{-x^2}\,\delta x \\ V &= \lim_{\delta x \to 0} \sum_{x=0}^{N} 2\pi x e^{-x^2}\,\delta x \\ &= \int_0^N 2\pi x e^{-x^2}\,dx \\ &= -\left[\pi e^{-x^2}\right]_0^N \\ &= \pi - \pi e^{-N^2} \text{ units}^3 \end{aligned}$$

<u>3 marks</u> : correct solution

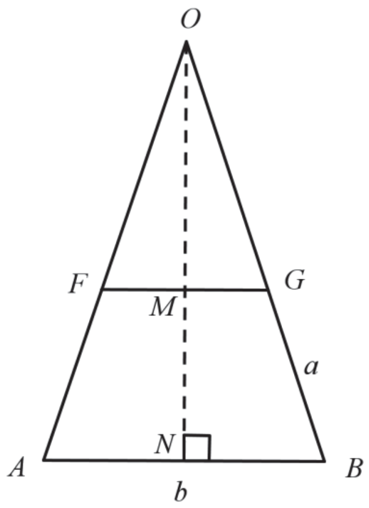
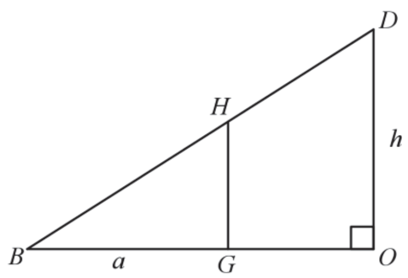
<u>2 marks</u> : substantially correct solution

<u>1 mark</u> : progress towards correct solution

(ii)

$$\begin{aligned} \lim_{N \to \infty} V &= \lim_{N \to \infty} \left(\pi - \pi e^{-N^2}\right) \\ &= \pi \text{ units}^3 \\ &\left(\text{note that } e^{-N^2} \to 0 \text{ as } N \to \infty\right) \end{aligned}$$

<u>1 mark</u> : correct solution

	Question 15 continued…	
(c)	(i) In base ΔOAB: $GB=a \qquad OB=r$ $NB=\frac{b}{2} \qquad OG=r-a$  $\frac{MG}{OG}=\frac{NB}{OB}$ $MG=\frac{NB.OG}{OB}$ $=\frac{b}{2}.\frac{r-a}{r}$ $FG=2MG=\frac{b(r-a)}{r}$ Also: $OD=h,\ OB=r,\ GB=a$  $\frac{GH}{GB}=\frac{OD}{OB}$ $GH=\frac{OD.GB}{OB}$ $=\frac{ah}{r}$ $V_S=FG.GH.\delta a$ $=\frac{b(r-a)}{r}\left(\frac{ah}{r}\right)\delta a$	<u>3 marks</u> : correct solution <u>2 marks</u> : substantially correct solution <u>1 mark</u> : progress towards correct solution
	(ii) $V=\int_0^r \frac{b(r-a)}{r}\left(\frac{ah}{r}\right)da$ $=\frac{bh}{r^2}\int_0^r a(r-a)\,da$ $=\frac{bh}{r^2}\int_0^r (ar-a^2)\,da$ $=\frac{bh}{r^2}\left[\frac{a^2r}{2}-\frac{a^3}{3}\right]_0^r$ $=\frac{bh}{r^2}\left[\left(\frac{r^3}{2}-\frac{r^3}{3}\right)-0\right]$ $=\frac{bh}{r^2}.\frac{r^3}{6} \qquad =\frac{1}{6}bhr$	<u>2 marks</u> : correct solution <u>1 mark</u> : substantially correct solution

Question 15 continued…

(iii) given $\angle AOB = \dfrac{2\pi}{n}$

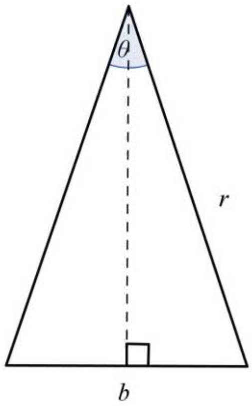
$$\text{ie} \quad \theta = \frac{2\pi}{n}$$

$$\frac{\theta}{2} = \frac{\pi}{n}$$

$$\text{now,} \quad \sin\frac{\theta}{2} = \frac{b}{2}.\frac{1}{r} \quad ✓$$

$$b = 2r\sin\frac{\theta}{2}$$

$$= 2r\sin\frac{\pi}{n}$$



$$V = \frac{1}{6}bhr \qquad (\text{from (ii)})$$

$$= \frac{1}{6}hr.2r\sin\frac{\pi}{n} \quad ✓$$

$$= \frac{1}{3}hr^2\sin\frac{\pi}{n}$$

$$V_n = \frac{1}{3}hr^2 n\sin\frac{\pi}{n}$$

<u>2 marks</u> : correct solution

<u>1 mark</u> : substantially correct solution

(iv)

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty}\frac{1}{3}r^2hn\sin\frac{\pi}{n}$$

$$= \frac{1}{3}r^2h\lim_{n\to\infty} n\sin\frac{\pi}{n}$$

$$= \frac{1}{3}r^2h\lim_{n\to\infty}\pi\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \quad ✓$$

$$\text{let } x = \frac{\pi}{n}; \quad \text{as } n\to\infty, \frac{\pi}{n}\to 0$$

✓

$$\text{so,} \quad \lim_{n\to\infty} V_n = \frac{1}{3}r^2h\pi\lim_{x\to 0}\frac{\sin x}{x}$$

$$= \frac{1}{3}\pi r^2 h$$

<u>2 marks</u> : correct solution

<u>1 mark</u> : substantially correct solution

Year 12	Mathematics Extension 2	Task 4 (Trial HSC) 2016
Question 16	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E9 communicates abstract ideas and relationships using appropriate notation and logical argument

Part	Solutions	Marking Guidelines
(a) (i)	$\angle RSA = \angle RAG$ (The angle between a tangent and a chord equals the angle at the circumference in the alternate segment of circle SAR) $= \alpha$ $\angle SPA = \angle RSA$ (The angle between a tangent and a chord equals the angle at the circumference in the alternate segment of circle $PBAS$) $= \alpha$	**Award 2** for correct solution **Award 1** for substantial progress towards solution
(ii)	$\angle AFP = \angle AGR = \beta$ (alternate angles are equal, $PS \parallel QR$) In ΔAFP and ΔRAG $\angle FPA (= \angle SPA) = \angle RAG$ (from (i)) $\angle AFP = \angle AGR$ (proved above) $\therefore \Delta AFP \parallel\!\mid \Delta RAG$ (equiangular) $\therefore \angle FAP = \angle GRA$ (matching angles in similar triangles are equal) $= \gamma$ $\angle PQA = \angle QRA$ (The angle between a tangent and a chord equals the angle at the circumference in the alternate segment of circle $RABQ$) $= \angle GRA$ $= \gamma$ $\therefore \angle FAP = \angle PQA$ Hence, FG is tangent to the circle through APQ by the converse of the angles in the alternate segment theorem.	**Award 3** for correct solution **Award 2** for substantial progress towards solution **Award 1** for limited progress towards solution
(b)	$(a-b)^2 = a^2 + b^2 - 2ab$ $(b-c)^2 = b^2 + c^2 - 2bc$ $(c-a)^2 = c^2 + a^2 - 2ca$ $2\left[a^2+b^2+c^2-(ab+bc+ca)\right] = (a-b)^2+(b-c)^2+(c-a)^2$ Now a, b and c are side lengths of the triangle and are all positive real numbers. $\therefore (a-b)^2 \geq 0$ and $(a-b)^2 = 0$ only if $a = b$ Hence if $a^2+b^2+c^2 = ab+bc+ca$ (given) then $(a-b)^2+(b-c)^2+(c-a)^2 = 0$ $\therefore (a-b)^2 = (b-c)^2 = (c-a)^2 = 0$ $\therefore a = b = c$ Therefore ΔABC is an equilateral triangle.	**Award 3** for correct solution **Award 2** for substantial progress towards solution **Award 1** for limited progress towards solution

(c) (i)	$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}^nC_k\left(\frac{1}{n}\right)^k$ $= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\left(\frac{1}{n}\right)^k$ $= \sum_{k=0}^{n}\frac{n(n-1)(n-2)...(n-k+1)}{n^k}.\frac{1}{k!}$	**Award 1** for correct solution
(ii)	$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\frac{n(n-1)(n-2)...(n-k+1)}{n^k}.\frac{1}{k!}$ $= \sum_{k=0}^{n}\frac{n}{n}\times\frac{(n-1)}{n}\times\frac{(n-2)}{n}\times...\times\frac{(n-k+1)}{n}.\frac{1}{k!}$ $= \frac{n!}{n!}+\frac{n!}{(n-1)!1!}.\frac{1}{n}+\frac{n!}{(n-2)!2!}.\frac{1}{n^2}+\frac{n!}{(n-3)!3!}.\frac{1}{n^3}+....+\frac{n!}{n!}.\frac{1}{n^n}$ $= 1+1+\frac{(n-1)}{n}.\frac{1}{2!}+\frac{(n-1)(n-2)}{n^2}.\frac{1}{3!}+....+\frac{1}{n!}$ As $n\to\infty$ then $\frac{n-1}{n}\to 1,\ \frac{n-2}{n}\to 1,\ \frac{n-3}{n}\to 1....$ $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n(n-1)(n-2)...(n-k+1)}{n^k}.\frac{1}{k!} =$ $= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{n}\times\frac{(n-1)}{n}\times\frac{(n-2)}{n}\times...\times\frac{(n-k+1)}{n}.\frac{1}{k!}$ $= \lim_{n\to\infty}\left(\frac{n!}{n!}+\frac{n!}{(n-1)!1!}.\frac{1}{n}+\frac{n!}{(n-2)!2!}.\frac{1}{n^2}+\frac{n!}{(n-3)!3!}.\frac{1}{n^3}+....+\frac{n!}{n!}.\frac{1}{n^n}\right)$ $= \lim_{n\to\infty}\left(1+1+\frac{(n-1)}{n}.\frac{1}{2!}+\frac{(n-1)(n-2)}{n^2}.\frac{1}{3!}+....+\frac{1}{n!}\right)$ $= 1+1+1.\frac{1}{2!}+1.\frac{1}{3!}+....+\frac{1}{n!}$ $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+... = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$	**Award 2** for correct solution **Award 1** for substantial progress towards solution
(iii)	Test the result for $n=3$ $\frac{1}{3!}<\frac{1}{2^{3-1}}$ or $\frac{1}{6}<\frac{1}{4}$. Therefore the result is true for $n=3$. Assume the result is true for $n=k$. $\frac{1}{k!}<\frac{1}{2^{k-1}}$ To prove the result is true for $n=k+1$. i.e. $\frac{1}{(k+1)!}<\frac{1}{2^{(k+1)-1}}<\frac{1}{2^k}$ $\text{LHS}=\frac{1}{(k+1)!}$ $=\frac{1}{(k+1)k!}$ $<\frac{1}{(k+1)2^{k-1}}$ Assumption for $n=k$ $<\frac{1}{2\times 2^{k-1}}$ $k+1>2$ as $n\geq 3$ $=\frac{1}{2^k}=$ RHS Thus if the result is true for $n=k$, it is true for $n=k+1$. It	**Award 3** for correct solution **Award 2** for proving the result true for $n=3$ and attempting to use the result of $n=k$ to prove the result for $n=k+1$. **Award 1** for proving the result true for $n=3$.

(iv)	has been shown true for $n=3$, hence true for $n=4$ and so on. From part (ii) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots = 2+\frac{1}{2}+\sum_{k=3}^{\infty}\frac{1}{k!}$ $< 2+\frac{1}{2}+\sum_{k=3}^{\infty}\frac{1}{2^{k-1}}$ $= 2+\frac{1}{2}+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\right)$ $= 2+\frac{1}{2}+\left(\frac{\frac{1}{2^2}}{1-\frac{1}{2}}\right)$ Limiting sum of GP $= 2+\frac{1}{2}+\frac{1}{2}=3$ **Multiple Choice Answers** 1. C 2. C 3. A 4. A 5. B 6. B 7. C 8. D 9. A 10. B	**Award 1** for correct solution